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EHP SPECTRA AND PERIODICITY. II: Λ-ALGEBRA MODELS

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ABSTRACT. The results of part I suggest that for small m, the Smith-Toda spectrum $V(m)$ can be approximated by spaces having universal mapping properties and interlocking fibrations. For each m, a Λ-algebra model representing the Bousfield-Kan E' term for these spaces is constructed, and all of the ideal results are proven on the chain level.

INTRODUCTION

The classical EHP sequences fit together to form the EHP spectral sequence. This filters the stable homotopy groups of spheres by "sphere of origin" and by its self-referential nature, creates a technology both for calculating homotopy groups and for analyzing results beyond the range in which we have a complete calculation.

In [G] we showed that the sphere spectrum is not the only spectrum with such a filtration, and gave evidence which suggests that for large enough primes, the spectrum $V(m)$, if it exists, should also have EHP sequences.

Constructing the fibrations suggested in [G] will not be an easy task for large values of m. It is our purpose here to present a Λ-algebra version of these EHP sequences in the spirit of Bousfield and Curtis [BC, C]. They describe short exact sequences of chain complexes:

$$H\colon 0 \longrightarrow \Lambda(2n) \xrightarrow{E} \Lambda(2n+1) \xrightarrow{H} \Lambda(2np+1) \longrightarrow 0,$$
$$H'\colon 0 \longrightarrow \Lambda(2n-1) \xrightarrow{E} \Lambda(2n) \xrightarrow{H'} \Lambda(2np-1) \longrightarrow 0$$

which induce long exact sequences in homology that correspond to the E^1 terms of the unstable Adams spectral sequences for the appropriate homotopy groups.

In addition to the EHP sequences, we produce short exact sequences corresponding to the sequences CMN, RCMN, and E^2 of [G] which relate adjacent values of m and establish the connection with periodicity. Thus, in the "world of ext", the conjectures in [G] are valid.

Throughout this paper we will be working at a prime $p > 2$. In 2.3 we define complexes $\Lambda_{(m)}(n)$ with $\Lambda_{(-1)}(n) = \Lambda(n)$. Our results are then summarized as

Theorem. *There are chain complexes* $\Lambda_{(m)}(n)$ *defined for* $n \geq 0$ *and* $m \geq -1$ *and inclusions* $\Lambda_{(m)}(n) \subset \Lambda_{(m)}(n+1)$ *such that*

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$$\Lambda_{(m)} = \bigcup_{n=1}^{\infty} \Lambda_{(m)}(n) = E(\tau_0, \dots, \tau_m) \widetilde{\otimes} \Lambda$$

where $E(\tau_0, \dots, \tau_m)$ *is the indicated subalgebra of the dual to the Steenrod algebra. Furthermore there are exact EHP sequences:*

$$\begin{array}{lllll} H: 0 \to \Lambda_{(m)}(2n) & \xrightarrow{E} \Lambda_{(m)}(2n+1) & \xrightarrow{H} \Lambda_{(m)}(2np + 2p^{m+1} - 1) & \to 0, \\ H': 0 \to \Lambda_{(m)}(2n-1) & \xrightarrow{E} \Lambda_{(m)}(2n) & \xrightarrow{H'} \Lambda_{(m)}(2np - 1) & \to 0 \end{array}$$

and $\Lambda_{(m)}(0)$ *is acyclic.*

There are also exact sequences:

$$\begin{array}{llll} E^2: & 0 \to \Lambda_{(m)}(2n-1) \xrightarrow{E^2} \Lambda_{(m)}(2n+1) & \xrightarrow{\varphi} \Lambda_{(m+1)}(2np-1) & \to 0, \\ CMN: & 0 \to \Lambda_{(m)}(2n+1) \xrightarrow{\iota} \Lambda_{(m+1)}(2n+1) & \xrightarrow{\gamma_m} \Lambda_{(m)}(2n + 2p^{m+1} + 1) & \to 0, \\ RCMN: & 0 \to \Lambda_{(m)}(2n+1) \xrightarrow{\iota_3} \Lambda_{(m+1)}(2n) & \xrightarrow{\gamma_m} \Lambda_{(m)}(2n + 2p^{m+1} - 1) & \to 0 \end{array}$$

corresponding to the double suspension sequence and the Cohen-Moore-Neisendorfer sequences of [G].

Finally $\Lambda_{(m)}$ is a differential algebra in such a way that

(a) $\Lambda_{(m)}(n)_\sigma \cdot \Lambda_{(m)}(n+\sigma) \subset \Lambda_{(m)}(n)$,

(b) The connecting homomorphism in the exact sequence *CMN* can be represented by a chain map of degree $q_m = 2(p^m - 1)$:

$$v_m : \Lambda_{(m-1)}(2n + 2p^m - 1) \to \Lambda_{(m-1)}(2n-1)$$

such that the diagram

$$\begin{array}{ccc} \Lambda_{(m-1)}(2n + 2p^m - 1) & \longrightarrow & \Lambda_{(m-1)}(2n+1) \\ \uparrow E^2 & \searrow^{v_m} & \uparrow E^2 \\ \Lambda_{(m-1)}(2n + 2p^m - 3) & \longrightarrow & \Lambda_{(m-1)}(2n-1) \end{array}$$

commutes where the horizontal maps are left composition with an element $v_m \in \Lambda_{(m-1)}(2r+1)_{q_m}$ for $r \geq 0$.

Note that property (a) generalizes a result of Harper and Miller in case $m = -1$ [HM, 1.17] and reflects the theory of compositions suggested in [G]. In property (b) the map v_m has degree q_m and is the algebraic analog of a v_m self-map.

This theorem summarizes the results in 2.3, 3.6, 3.7, 4.1, 4.2, 5.1, 5.2, 6.3, 6.5 and 6.6.

One striking corollary concerns v_m periodic homotopy:

Corollary 5.2.

$$v_m^{-1}\Lambda_{(m-1)}(2n-1) \cong v_m^{-1}\Lambda_{(m-1)}(2n+1) \cong v_m^{-1}(E(\tau_0, \dots, \tau_{m-1}) \widetilde{\otimes} \Lambda).$$

This generalizes one of the main results of [HM].

In §7, we introduce the category of unstable right Λ modules, which we take to be an approximation to the unstable homotopy category. The properties of this category reinforce various delooping conjectures of [G].

In §8, we make explicit, the EHP spectral sequence obtained for calculating $H_*(\Lambda_{(m)}) \cong \mathrm{Ext}_A(H^*(V(m)); \mathbf{Z}_p)$.

1

We begin by recalling the Bousfield-Kan construction [BK]. This is an explicit chain complex for calculating $\mathrm{Ext}_A(M^*; \mathbf{Z}_p)$ where A is the mod p Steenrod algebra and M is a right A module. It is obtained by putting a twisted differential (see 1.4) on the tensor product of M and Λ, where Λ is the Λ algebra.

The Λ algebra is filtered by $\Lambda(n)$ corresponding to the sphere[1] S^n. We will construct a filtration of the Bousfield-Kan construction $M \tilde{\otimes} \Lambda$ induced by a filtration on M. A particularly important special case applies to submodules of A_* (the dual of the Steenrod algebra).

Definition 1.1. Let M be a right A module. A decreasing filtration $F_k M$ will be called useful if

(a) $(F_k M)\mathscr{P}^n \subset F_k M$,

(b) $(F_k M)\beta \subset F_{k-1} M$,

(c) If $(F_k M - F_{k+1} M)_r \beta \neq 0$, $k + r$ is even.

Example 1.2. Let $M = A_*$, the dual of the Steenrod algebra considered as a right A module. Let $F_k M = \bigoplus \tau_{i_1} \cdots \tau_{i_k} A_*$ where the sum is over all sequences of integers of length k and $\tau_i \in A_*$ is the standard exterior generator of dimension $2p^i - 1$. Using the comultiplication in A_* one easily sees that $\tau_i \mathscr{P}^n = \epsilon \tau_{i-1}$ where $\epsilon = 1$ if $n = p^{i-1}$ and 0 otherwise. Thus a is satisfied. Likewise $\xi_i \beta = 0$ and $\tau_i \beta = 0$ if $i > 0$ while $\tau_0 \beta = 1$. Thus b is satisfied. Finally $(F_k M)_r = (F_{k+1} M)_r$ if $k + r$ is odd, and c is satisfied. This will be called the Bockstein filtration since the dual filtration of A is by the number of Bocksteins.

Proposition 1.3. *If $F_k M$ is a useful filtration of M and $N \subset M$ is a submodule, the induced filtration on N given by $F_k N = N \cap F_k M$ is useful.*

This gives a useful filtration on the dual of any cyclic left A module. Of course another useful filtration is obtained by setting $F_k M = M$ for all k.

The Bousfield-Kan construction $M \tilde{\otimes} \Lambda$ provides us with a differential on the tensor product $M \otimes \Lambda$. The differential is defined by the formula

$$(1.4) \qquad \partial(x \otimes \nu) = (-1)^{|x|} \sum_{i>0} x\mathscr{P}^i \otimes \lambda_i \nu + \sum_{i \geq 0} x\beta\mathscr{P}^i \otimes \mu_i \nu + (-1)^{|x|} x \otimes \partial \nu$$

for $x \in M$ and $\nu \in \Lambda$ where $|x|$ is the dimension of x.

Suppose now that M has a useful filtration. We define a subspace $M(2n-1) \subset M \tilde{\otimes} \Lambda$ as follows:

$$M(2n-1) = \bigcup (F_k M)_r \otimes \Lambda(2n - 1 + k + r).$$

We now define the depth of a useful filtration by the formula:

$$d(M) = \max\{2pi - k - r \mid (F_k M)_r \mathscr{P}^i \neq 0\}.$$

[1] In case n is even, $\Lambda(n)$ corresponds to the "homotopy theorists even sphere"; i.e., the $p-1$ cell complex $J_{p-1}(S^n)$.

Proposition 1.5. *If* $2n-1 \geq d(M)$, $M(2n-1)$ *is a subcomplex.*

To prove this we require a result of Harper and Miller [HM, 1.18] which we will use frequently in the sequel.

Proposition 1.6. *The subcomplexes* $\Lambda(n) \subset \Lambda$ *satisfy*

$$\lambda_k \Lambda(n) \subset \Lambda(n-kq) \qquad \text{if } n \geq 2pk > 0,$$
$$\mu_k \Lambda(2n+1) \subset \Lambda(2n-kq-1) \quad \text{if } n > pj \geq 0.$$

Proof of Proposition 1.5. It suffices to show that if $x \in (F_k M)_r$ and $\nu \in \Lambda(2n-1+k+r)$ then $x\mathscr{P}^i \otimes \lambda_i \nu$ and $x\beta\mathscr{P}^i \otimes \mu_i \nu$ both belong to $M(2n-1)$ when $2n-1 \geq d(M)$. Now $x\mathscr{P}^i \in (F_k M)_{r-iq}$ and

$$\lambda_i \nu \in \lambda_i \Lambda(2n-1+k+r) \subseteq \Lambda(2n-1+k+r-iq)$$

by 1.6 if $2n-1+k+r \geq 2pi > 0$. This holds if $2n-1 \geq d(M)$. Consequently $x\mathscr{P}^i \otimes \lambda_i \nu \in M(2n-1)$. Likewise $x\beta\mathscr{P}^i \in (F_{k-1}M)_{r-iq-1}$ and $\mu_i \nu \in \mu_i \Lambda(2n-1+k+r) \subseteq \Lambda(2n-3+k+r-iq)$ if $k+r$ is even and $n-1+\frac{k+r}{2} > pi \geq 0$ by 1.6. If k is maximal with $x \in F_k M$ and $x\beta \neq 0$, $k+r$ must be even by 1.1(c). Finally, since $x\beta \in (F_{k-1}M)_{r-1}$, $(x\beta)\mathscr{P}^i \neq 0$ implies that $2n-1 \geq d(M) \geq 2pi-(k-1)-(r-1) = 2pi-k-r+2$, so $2n-2+k+r > 2pi$ completing the proof.

The following result will be proven in the next section.

Proposition 1.7. *If* $M \subset E(\tau_0, \tau_1, \ldots) \subset A_*$ *with the Bockstein filtration,* $d(M)=0$.

2

The goal of this section is to study the right A module $HV(m)$ which could occur as the homology of a Smith-Toda complex $V(m)$. We will show that the A module structure is unique and isomorphic to the appropriate submodule of A_*. The A module structure of submodules of A_* is then noted.

Let $[m] = \{0, 1, \ldots, m\}$ and $[-1] = \phi$. For $S \subset [m]$ write $Q^S = Q^{i_1} \cdots Q^{i_k}$ where $S = \{i_1 \cdots i_k\}$ and $i_1 < i_2 < \cdots < i_k$. Let H^* be a left A module which has as a basis all $Q^S u$, $S \subset [m]$ where $|u| = 0$.

Lemma 2.1. *There is a unique left* A *module structure on* H^*. *With this structure* $\mathscr{P}^I u = 0$ *for each pure Steenrod operation* $\mathscr{P}^I$ *with* $I \neq 0$. *Let* $HV(m)$ *be the vector space dual to* H^*. *Then the image of* $HV(m)$ *in* A_* *is the subalgebra* $E(\tau_0, \ldots, \tau_m)$.

Proof. To see that $\mathscr{P}^I u = 0$ for all $I \neq 0$ it suffices to prove that $\mathscr{P}^{p^n} u = 0$ if $n \geq 0$. If not, $\mathscr{P}^{p^n} u = Q^S u$ for $S = \{i_1 \cdots i_k\}$ and consequently $2p^n(p-1) = 2(p^{i_1} + \cdots + p^{i_k}) - k$. Since $i_1 < i_2 < \cdots < i_k$, $i_k \geq k-1$. Also $k = 2t$ so we get

$$t + p^n(p-1) = p^{i_1} + \cdots + p^{i_k}.$$

Studying the p-adic expansions of this equation one sees that $t \geq p^{i_k - 1}$ and hence that $k \geq 2p^{k-2}$ which implies that $k = 2$. Again looking at p-adic expansions we see that this is impossible.

Now consider the left A module map $A \to H^*$ which sends 1 to u. This is an epimorphism and determines a monomorphism $HV(m) \subset A_*$. Since

$\mathscr{P}^I u = 0$ for each pure Steenrod operation P^I, $HV(m) \subseteq E(\tau_0, \tau_1, \dots)$. For dimensional reasons $HV(m) = E(\tau_0, \dots, \tau_m)$. Consequently the right A module structure on $HV(m)$ is determined.

Our task now is to clarify the A module structure on $E(\tau_0, \dots)$. To this end we define some notation. For $S = \{i_1, \dots, i_k\}$ write $\tau_S = \tau_{i_1} \cdots \tau_{i_k}$ if $i_1 < i_2 < \cdots < i_k$. Let $S' = \{i+1 | i \in S\}$ and $(S, T) = (S - T') \cup T$ if $T' \subset S$. Let

$$\tau(S, T) = \begin{cases} \tau_{(S,T)} & \text{if } (S - T') \cap T = \phi, \\ 0 & \text{otherwise} \end{cases}$$

(this implies that $\tau(S, T) = \pm\tau_{S-T'}\tau_T$). Now write $|S|$ for the number k of elements in S and put $p^S = p^{i_1} + \cdots + p^{i_k}$ and $p^\phi = 0$. Thus $p \cdot p^T = P^{T'}$. In evaluating the right A module action it is important to observe that the following unusual Cartan formula for the Bockstein [BK, 11.2]

$$(xy)\beta = (x\beta)y + (-1)^{|x|}x(y\beta).$$

Proposition 2.2.

$$\tau_S\beta = \begin{cases} \tau(S - \{0\}) & \textit{if } 0 \in S, \\ 0 & \textit{if } 0 \notin S, \end{cases}$$

$$\tau_S\mathscr{P}^n = \begin{cases} \tau(S, T) & \textit{if } n = p^T, \\ 0 & \textit{otherwise, } n \neq 0. \end{cases}$$

Proof. Since $\tau_i\beta = 0$ except when $i = 0$ and $\tau_0\beta = 1$ the Cartan formula gives the first equation. Since $\tau_i\mathscr{P}^n = \epsilon\tau_{i-1}$ where $\epsilon = 1$ if $n = p^{i-1}$ and $\epsilon = 0$ otherwise, an application of the Cartan formula to τ_S shows that this is only nonzero when n is a sum of distinct powers of p and $\tau_S\mathscr{P}^{p^T} \neq 0$ only if $T' \subset S$. In this case each element in $S \cap T'$ is decreased by one. The resulting sequence is still in numerical order although there may be repetitions, in which case that term is 0. This happens precisely when $(S - T') \cap T \neq \phi$, in which case $\tau(S, T) = 0$.

Proof of Proposition 1.7. If $\tau_S\mathscr{P}^n \neq 0$, $n = p^T$ and $T' \subset S$. Thus $2np-k-r = 2p^{T'} - |S| - (2p^S - |S|) = 2(p^{T'} - p^S) \leq 0$.

Finally we describe $M(2n-1)$ in case $M = E(\tau_0, \dots, \tau_m)$ in complete detail for future reference.

Proposition 2.3. *Let* $\Lambda_{(m)}(2n-1)$ *be the subspace of* $E(\tau_0, \dots, \tau_m)\tilde{\otimes}\Lambda$ *defined by*

$$\Lambda_{(m)}(2n-1) = \sum_{S \subset [m]} \tau_S \otimes \Lambda(2n - 1 + 2p^S).$$

Then $\Lambda_{(m)}(2n-1)$ *is a subcomplex for* $n \geq 1$ *with differential given by the formula*

$$\begin{aligned} \partial(\tau_S \otimes \nu) = (-1)^{|S|} &\sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(S, T) \otimes \lambda_{p^T}\nu \\ &+ \sum_{\{0\} \cup T' \subset S} \tau(S - \{0\}, T) \otimes \mu_{p^T}\nu + (-1)^{|S|}\tau_S \otimes \partial\nu \end{aligned}$$

for $S \subset [m]$ *and* $\nu \in \Lambda(2n-1+2p^S)$. *Note that* $\Lambda_{(-1)}(2n-1) = \Lambda(2n-1)$.

It is also interesting, in the light of [HM] to consider the submodule of A_* dual to A_1. $(A_1)_*$ has a unique A module structure and has a $\mathbf{Z}_p$ basis consisting of $\tau_0^{\epsilon_1}\tau_1^{\epsilon_2}\xi_1^k$ where $0 \le \epsilon_i \le 1$ and $0 \le k \le p-1$. Using the Bockstein filtration we easily get $d((A_1)_*) = q$ and consequently there are subcomplexes $A_1(2n-1)$ of $(A_1)_*\tilde{\otimes}\Lambda$ for each $n \ge p$. Furthermore $\widetilde{F}(n+1) \subset A_1(2np^2-1) \subset F(n+1)$ where $\widetilde{F}(n+1)$ and $F(n+1)$ are the complexes defined in [HM]. $A_1(2n-1)$ is the same form as the Bousfield-Kan construction on a space X filtered by subspaces $V_1(2n-1) = X_0 \subset X_1 \subset X_2 \subset X_{p-1}$ with fibrations $X_{i-1} \to X_i \to V_{(1)}(2n+iq-1)$. (See [G] for notation.)

3

We begin this section by analyzing the Hopf invariants in the Λ algebra. We will prove some technical lemmas about their behavior on composites generalizing to odd primes a result of Singer [S]. We will use these results to define Hopf invariants for $\Lambda_{(m)}$.

Let us recall the definition.

Definition 3.1. Let $\nu \in \Lambda(2n+1)$. By expanding in terms of admissible monomials, there is a unique expression

$$\nu = \nu_1 + \mu_n\nu_2 + \lambda_n\nu_3$$

with $\nu_1 \in \Lambda(2n-1)$, $\nu_2 \in \Lambda(2np+1)$, and $\nu_3 \in \Lambda(2np-1)$. If $n = 0$, $\nu_1 = \nu_3 = 0$. We write $H_n(\nu) = \nu_2$ and $H_n'(\nu) = \nu_3$. This defines maps of degree $-nq$ and $-(nq-1)$ respectively.

$$\Lambda(2n+1) \xrightarrow{H_n} \Lambda(2np+1), \quad \Lambda(2n+1) \xrightarrow{H_n'} \Lambda(2np-1).$$

ν_1 is completely determined by the formula $\nu_1 = \nu - \mu_n H_n(\nu) - \lambda_n H_n'(\nu)$.

Lemma 3.2. H_n *is a chain map while* $H_n'\partial = -\partial H_n' - \mu_0 H_n$. *Consequently there are short exact sequences of chain complexes*:

$$\begin{array}{lccccccccc} H: & 0 & \to & \Lambda(2n) & \xrightarrow{E} & \Lambda(2n+1) & \xrightarrow{H_n} & \Lambda(2np+1) & \to & 0, \\ H': & 0 & \to & \Lambda(2n-1) & \xrightarrow{E} & \Lambda(2n) & \xrightarrow{H_n'} & \Lambda(2np-1) & \to & 0. \end{array}$$

Proof. Differentiate the formula in 3.1.

The Harper-Miller result (1.6) allows us to define homomorphisms via left composition:

$$\begin{array}{lcll} \Lambda(2n+1) & \xrightarrow{\lambda_k} & \Lambda(2n-kq+1) & \text{if } kp \le n, \\ \Lambda(2n+1) & \xrightarrow{\mu_k} & \Lambda(2n-kq-1) & \text{if } kp+1 \le n. \end{array}$$

Proposition 3.3. *Suppose* $\nu \in \Lambda(2n+1)$. *If* $kp \le n$ *we have*

(a) $H_{n-k(p-1)}(\lambda_k\nu) = \lambda_{kp}H_n(\nu)$,

(b) $H_{n-k(p-1)}'(\lambda_k\nu) = \epsilon - \mu_{kp}H_n(\nu) - \lambda_{kp}H_n'(\nu)$ *where*

$$\epsilon = \begin{cases} 0 & \textit{if } kp < n, \\ \nu & \textit{if } kp = n. \end{cases}$$

If $kp+1 \le n$ *we have*

(c) $H'_{n-k(p-1)-1}(\mu_k\nu) = 0$,

(d) $H_{n-k(p-1)-1}(\mu_k\nu) = \epsilon - \mu_{kp+1}H_n(\nu) - \lambda_{kp+1}H'_n(\nu)$ *where*

$$\epsilon = \begin{cases} 0 & \text{if } kp+1 < n, \\ \nu & \text{if } kp+1 = n. \end{cases}$$

Each of these results can be expressed, somewhat more conceptually, as a commutative diagram. For example, a) *can be written*

$$\begin{array}{ccc} \Lambda(2n+1) & \xrightarrow{\lambda_k} & \Lambda(2n-kq+1) \\ \downarrow H_n & & \downarrow H_{n-k(p-1)} \\ \Lambda(2np+1) & \xrightarrow{\lambda_{kp}} & \Lambda(2np-kpq+1). \end{array}$$

Proof. We consider first the cases (a) and (b). Write $\nu = \nu_1 + \mu_n\nu_2 + \lambda_n\nu_3$ as in 3.1 and consider the composition.

$$\lambda_k\nu = \lambda_k\nu_1 + \lambda_k\mu_n\nu_2 + \lambda_k\lambda_n\nu_3 \in \Lambda(2n-kq+1).$$

We look at the last term first. $\lambda_k\lambda_n$ is expressible as a sum of terms of the form $\lambda_{n-k(p-1)-j}\lambda_{kp+j}$ with $j < n-kp$. If the inequalities in 1.6 are satisfied we can write

$$\begin{aligned} \lambda_{n-k(p-1)-j}\lambda_{kp+j}\nu_3 &\in \lambda_{n-k(p-1)-j}\lambda_{kp+j}\Lambda(2np-1) \\ &\subset \lambda_{n-k(p-1)-j}\Lambda(2np-(kp+j)q-1) \\ &\subset \Lambda(2n-kq-1). \end{aligned}$$

Such terms do not contribute to either Hopf invariant. The requisite inequalities are:

(1) $2(kp+j)p \le 2np-1$ which holds since $j < n-kp$

(2) $2(n-k(p-1)-j) \le 2np-(kp+j)q-1$ which holds when $j \ge 1$.

The term with $j = 0$ has coefficient -1, so $\lambda_k\lambda_n\nu_3 \equiv -\lambda_{n-k(p-1)}\lambda_{kp}\nu_3$ (modulo $\Lambda(2n-kq-1)$). Similarly we may write $\mu_k\mu_n$ as a sum of terms of either the form $\lambda_{n-k(p-1)-j}\mu_{pk+j}$ or $\mu_{n-k(p-1)-j}\lambda_{pk+j}$ with $j < n-kp$. A similar analysis shows that the only terms not in $\Lambda(2n-kq-1)$ correspond to $j = 0$.

Suppose now that $kp < n$. Then $\lambda_k\mu_n\nu_2 \equiv (-\lambda_{n-k(p-1)}\mu_{kp} - \mu_{n-k(p-1)}\lambda_{kp})\nu_2$ (modulo $\Lambda(2n-kq-1)$). Also $\lambda_k\nu_1 \in \Lambda(2n-kq-1)$ by 1.6. We then have

$$\lambda_k\nu \equiv -\lambda_{n-k(p-1)}(\mu_{kp}\nu_2 + \lambda_{kp}\nu_3) + \mu_{n-k(p-1)}\lambda_{kp}\nu_2$$

modulo $\Lambda(2n-kq-1)$. This completes the proof of (a) and (b) in case $kp < n$. If $kp = n$, $\lambda_k\nu = \lambda_k\nu_1 + \mu_k\lambda_{kp}\nu_2$ since $\lambda_k\lambda_{kp} = 0$ and $\lambda_k\mu_{kp} = \mu_k\lambda_{kp}$. Thus $H_k(\lambda_k\nu) = \lambda_{kp}H_{kp}(\nu)$ and $H'_k(\lambda_k\nu) = \nu_1 = \nu - \lambda_{kp}H'_{kp}(\nu) - \mu_{kp}H_{kp}(\nu)$.

The cases of (c) and (d) are similar. If $n > kp+1$,

$$\mu_k\nu \equiv -\mu_{n-k(p-1)-1}\mu_{kp+1}\nu_2 - \mu_{n-k(p-1)-1}\lambda_{kp+1}\nu_3 \quad (\text{modulo } \Lambda(2n-kq-3)).$$

If $n = kp+1$, $\mu_k\nu = \mu_k\nu_1$ since $\mu_k\mu_{kp+1} = 0 = \mu_k\lambda_{kp+1}$ so $H_k(\mu_k\nu) = \nu_1 = \nu - \mu_nH_n(\nu) - \lambda_nH'_n(\nu)$.

At this point we will introduce intermediate complexes $\Lambda_{(m)}(2n)$ for $n \ge 0$, $m \ge 0$. These correspond to the "even spheres" in the EHP filtration. First set

$$\epsilon_m(S) = \begin{cases} 0, & m \in S, \\ 2, & m \notin S, \end{cases}$$

and then

$$\Lambda_{(m)}(2n) = \sum_{S \subset [m]} \tau_S \otimes \Lambda(2n-1+2p^S+\epsilon_m(S)).$$

Lemma 3.4. *Each of the modules on the left are subcomplexes of the complexes on the right*

$$\Lambda_{(m)}(2n+1) \xrightarrow{\iota} \Lambda_{(m+1)}(2n+1),$$
$$\Lambda_{(m)}(2n) \xrightarrow{\iota_1} \Lambda_{(m)}(2n+1),$$
$$\Lambda_{(m)}(2n-1) \xrightarrow{\iota_2} \Lambda_{(m)}(2n),$$
$$\Lambda_{(m)}(2n+1) \xrightarrow{\iota_3} \Lambda_{(m+1)}(2n).$$

Proof. Since $\iota = \iota_1\iota_3$ we need only consider the last three cases. By definition, they are all submodules. It remains to show that $\Lambda_{(m)}(2n)$ is closed under ∂. Suppose first that $\epsilon_m(S) = 2$, and $\partial(\tau_S \otimes \nu) = \sum \tau(i) \otimes \nu_i$. Then $\epsilon_m(\tau(i)) = 2$ so this case is clear. The case $\epsilon_m(S) = 0$ is obvious in case $n \geq 1$. Consider then the case $n = 0$ and $\epsilon_m(S) = 0$. Referring to the proof of 1.7, we see that the problem terms correspond to $T' = S$ or $\{0\} \cup T' = S$ and $m \in (S, T)$ (or $m \in (S - \{0\}, T)$). This cannot occur.

These complexes will play a crucial role in the EHP development of the next section. For $\tau_S \otimes \nu \in \Lambda_{(m)}(2n+1)$ define a function $\varphi(\tau_S \otimes \nu)$ by

$$\varphi(\tau_S \otimes \nu) = (-1)^{|S|}\{\tau_{S'} \otimes H'_{n+p^S}(\nu)\} + \tau(\{0\} \cup S') \otimes H_{n+p^S}(\nu).$$

Theorem 3.5. *If* $n > 0$, φ *defines a chain map* (*in the graded sense*) *of degree* $-(nq-1)$:

$$\Lambda_{(m)}(2n+1) \xrightarrow{\varphi} \Lambda_{(m+1)}(2np-1)$$

with $\varphi(\Lambda_{(m)}(2n)) \subset \Lambda_{(m)}(2np-1)$.

If $n = 0$, φ *defines a degree* 1 *map*

$$\Lambda_{(m)}(1) \to \Lambda_{(m+1)}(0)$$

with $\varphi(\Lambda)_{(m)}(0)) \subset \Lambda_{(m)}(0)$ *such that* $\varphi\partial + \partial\varphi = \iota_3$ *in positive dimensions.*

Corollary 3.6. $\Lambda_{(m)}(0)$ *is acyclic. That is,*

$$H_r(\Lambda_{(m)}(0)) = \begin{cases} \mathbf{Z}_p, & r = 0, \\ 0, & r > 0. \end{cases}$$

Proof of Corollary 3.6. Let $D = \varphi|_{\Lambda_{(m)}(0)}$. Then $\varphi D + D\varphi = 1 - \eta$ where η is augmentation.

Proof of Theorem 3.5.

$$H'_{n+p^S}(\nu) \in \Lambda(2np - 1 + 2p^{S'}) \quad \text{and} \quad H_{n+p^S}(\nu) \in \Lambda(2np - 1 + 2p^{S' \cup \{0\}}).$$

These integers are positive if $n > 0$ so φ is a well-defined homomorphism in this case. In case $n = 0$ the only difficulty occurs when $S = \phi$. In this case $H'_0 = 0$.

It is left to evaluate $\varphi\partial$ and $\partial\varphi$.

$$\begin{aligned}\varphi(\partial(\tau_S \otimes \nu)) = & \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(S, T)' \otimes H'_{n+p^{(S,T)}}(\lambda_{p^T}\nu) \\ & + (-1)^{|S|} \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(\{0\} \cup (S, T)') \otimes H_{n+p^{(S,T)}}(\lambda_{p^T}\nu) \\ & + (-1)^{|S|-1} \sum_{\{0\} \cup T' \subset S} \tau(S - \{0\}, T)' \otimes H'_{n+p^{(S-\{0\},T)}}(\mu_{p^T}\nu) \\ & + \sum_{\{0\} \cup T' \subset S} \tau(\{0\} \cup (S - \{0\}, T)') \otimes H_{n+p^{(S-\{0\},T)}}(\mu_{p^T}\nu) \\ & + \tau_{S'} \otimes H'_{n+p^S}(\partial\nu) + (-1)^{|S|}\tau(\{0\} \cup S') \otimes H_{n+p^S}(\partial\nu).\end{aligned}$$

Now assume $n \geq 1$ and apply 3.3 with $n + p^S$ for n and p^T for k. This gives

$$\begin{aligned}\varphi(\partial(\tau_S \otimes \nu)) = & \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(S, T)' \otimes (-\mu_{p^{T'}} H_{n+p^S}(\nu) - \lambda_{p^{T'}} H_{n+p^S}(\nu)) \\ & + (-1)^{|S|} \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(\{0\} \cup (S, T)') \otimes \lambda_{p^{T'}} H_{n+p^S}(\nu) \\ & + \sum_{\{0\} \cup T' \subset S} \tau(\{0\} \cup (S - \{0\}, T)') \\ & \otimes (-\mu_{p^{T'}+1} H_{n+p^S}(\nu) - \lambda_{p^{T'}+1} H'_{n+p^S}(\nu)) \\ & + \tau_{S'} \otimes H'_{n+p^S}(\partial\nu) + (-1)^{|S|}\tau(\{0\} \cup S') \otimes H_{n+p^S}(\partial\nu).\end{aligned}$$

In case $n = 0$ we must exclude the terms from the first and third sum in which $T' = S$ ($\{0\} \cup T' = S$) and add $\tau_S \otimes \nu_1$.

We now collect terms using

$$\tau(S', W) = \begin{cases} \tau(S, T)' & \text{if } W = T', \\ \tau(\{0\} \cup (S - \{0\}, T)') & \text{if } W = \{0\} \cup T' \end{cases}$$

to obtain

$$\begin{aligned}\varphi(\partial(\tau_S \otimes \nu)) = & \sum_{\substack{W \subset S \\ W \neq \phi}} \tau(S', W) \otimes (-\mu_{p^W} H_{n+p^S}(\nu) - \lambda_{p^W} H'_{n+p^S}(\nu)) \\ & +(-1)^{|S|} \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(\{0\} \cup (S, T)') \otimes \lambda_{p^{T'}} H_{n+p^S}(\nu) - \tau_{S'} \otimes \partial H'_{n+p^S}(\nu) \\ & + (-1)^{|S|}\tau(\{0\} \cup S') \otimes \partial H_{n+p^S}(\nu) - \tau_{S'} \otimes \mu_0 H_{n+p^S}(\nu).\end{aligned}$$

On the other hand,

$$\begin{aligned}\partial(\varphi(\tau_S\otimes\nu)) &= (-1)^{|S|}\partial(\tau_{S'}\otimes H'_{n+p^S}(\nu)) + \partial(\tau(\{0\}\cup S')\otimes H_{n+p^S}(\nu))\\ &= \sum_{\substack{T\subset S\\ T\neq\phi}}\tau(S', T)\otimes\lambda_{p^T}H'_{n+p^S}(\nu) + \tau_{S'}\otimes\partial H'_{n+p^S}(\nu)\\ &\quad + (-1)^{|S|+1}\sum_{\substack{T\subset S\\ T\neq\phi}}\tau(S'\cup\{0\}, T)\otimes\lambda_{p^T}H_{n+p^S}(\nu)\\ &\quad + \sum_{T\subset S}\tau(S', T)\otimes\mu_{p^T}H_{n+p^S}(\nu)\\ &\quad + (-1)^{|S|+1}\tau(\{0\}\cup S')\otimes\partial H_{n+p^S}(\nu).\end{aligned}$$

These are clearly equal with the opposite sign. Thus $\varphi\partial = -\partial\varphi$ when $n \geq 1$. When $n = 0$ all the terms of $\varphi\partial + \partial\varphi$ cancel except those corresponding to $W = S$ in $\varphi\partial$ and $T = S$ in the first and fourth terms of $\partial\varphi$. The resulting sum is

$$\tau_S\otimes\nu_1 + \tau_S\otimes\lambda_{p^S}H'_{p^S}(\nu) + \tau_S\otimes\mu_{p^S}H_{p^S}(\nu)$$

which is $\tau_S\otimes\nu$.

Theorem 3.7. *If* $n \geq 1$ *there is an exact sequence of chain complexes*

$$E^2 : 0 \to \Lambda_{(m)}(2n-1) \xrightarrow{E^2} \Lambda_{(m)}(2n+1) \xrightarrow{\varphi_n} \Lambda_{(m+1)}(2np-1) \longrightarrow 0.$$

Proof. Clearly E^2 is 1-1 and $\varphi\cdot E^2 = 0$. If $\varphi(\sum\tau_{S_i}\otimes\nu_i) = 0$, $H_{n+p^{S_i}}(\nu_i) = 0 = H'_{n+p^{S_i}}(\nu_i)$ since the elements $\{\tau_{S'_i}, \tau(\{0\}\cup S'_i)\}$ are independent. Thus each ν_i is a double suspension and $\sum\tau_{S_i}\otimes\nu_i$ is in the image of E^2. Since φ is clearly onto we are done.

4

In this section we will describe the EHP sequences and the Cohen-Moore-Neisendorfer exact sequences. In all there are five interlocking short exact sequences of chain complexes giving five long exact sequences in homology. These are direct analogues of geometric fibrations proposed in [G].

Proposition 4.1. *There are short exact sequences*:

$$\begin{aligned} CMN:\quad & 0 \to \Lambda_{(m)}(2n+1) \xrightarrow{\iota} \Lambda_{(m+1)}(2n+1) \xrightarrow{\gamma_m} \Lambda_{(m)}(2n+2p^{m+1}+1) \to 0,\\ RCMN:\quad & 0 \to \Lambda_{(m)}(2n+1) \xrightarrow{\iota_3} \Lambda_{(m+1)}(2n) \xrightarrow{\gamma_m} \Lambda_{(m)}(2n+2p^{m+1}-1) \to 0\end{aligned}$$

where $n \geq 0$ *and* π_m *has degree* $2p^{m+1}-1$ *such that the diagram*

$$\begin{array}{ccccccccc} 0 \to & \Lambda_{(m)}(2n-1) & \xrightarrow{\iota} & \Lambda_{(m+1)}(2n-1) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1}-1) & \to 0\\ & \downarrow E^2 & & \downarrow E & & \| & \\ 0 \to & \Lambda_{(m)}(2n+1) & \xrightarrow{\iota_3} & \Lambda_{(m+1)}(2n) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1}-1) & \to 0\\ & \| & & \downarrow E & & \downarrow E^2 & \\ 0 \to & \Lambda_{(m)}(2n+1) & \xrightarrow{\iota} & \Lambda_{(m+1)}(2n+1) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1}+1) & \to 0\end{array}$$

commutes.

Proof. Define

$$\gamma_m(\tau_S \otimes \nu) = \begin{cases} (-1)^{|T|}\tau_T \otimes \nu & \text{if } S = T \cup \{m+1\}, \\ 0 & \text{if } m+1 \notin S. \end{cases}$$

All of the assertions are easy to check.

Proposition 4.2. *There are short exact EHP sequences:*

$$\begin{array}{llll} H: 0 \to \Lambda_{(m)}(2n) & \xrightarrow{E} \Lambda_{(m)}(2n+1) & \xrightarrow{H} \Lambda_{(m)}(2np+2p^{m+1}-1) & \to 0 \\ H': 0 \to \Lambda_{(m)}(2n-1) & \xrightarrow{E} \Lambda_{(m)}(2n) & \xrightarrow{H'} \Lambda_{(m)}(2np-1) & \to 0 \end{array}$$

where H and H' have degrees $-(2p^{m+1}+nq-2)$ and $-(nq-1)$ respectively. Furthermore the diagrams:

(I)

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 \to & & \Lambda_{(m)}(2n-1) & \xrightarrow{\iota} & \Lambda_{(m+1)}(2n-1) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1)}-1) \to 0 \\ & & \downarrow E^2 & & \downarrow E & & \Vert \\ 0 \to & & \Lambda_{(m)}(2n+1) & \xrightarrow{\iota_3} & \Lambda_{(m+1)}(2n) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1}-1) \to 0 \\ & & \downarrow \varphi_n & & \downarrow H' & & \\ & & \Lambda_{(m+1)}(2np-1) & = & \Lambda_{(m+1)}(2np-1) & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

(II)

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 \to \Lambda_{(m)}(2n+1) & \xrightarrow{\iota_3} & \Lambda_{(m+1)}(2n) & \xrightarrow{\gamma_m} & \Lambda_{(m)}(2n+2p^{m+1}-1) \to 0 \\ \Vert & & \downarrow E & & \downarrow E^2 \\ 0 \to \Lambda_{(m)}(2n+1) & \xrightarrow{\iota} & \Lambda_{(m+1)}(2n+1) & \to & \Lambda_{(m)}(2n+2p^{m+1}+1) \to 0 \\ & & \downarrow E & & \downarrow \varphi \\ & & \Lambda_{(m+1)}(2np+2p^{m+2}-1) & = & \Lambda_{(m+1)}(2np+2p^{m+2}-1) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

(III)

$$\begin{array}{ccccc} & & 0 & & 0 \\ & & \downarrow & & \downarrow \\ 0 \to \Lambda_{(m)}(2n-1) & \xrightarrow{E} & \Lambda_{(m)}(2n) & \xrightarrow{H'} & \Lambda_{(m)}(2np-1) \to 0 \\ \Vert & & \downarrow E & & \downarrow \\ 0 \to \Lambda_{(m)}(2n-1) & \xrightarrow{E^2} & \Lambda_{(m)}(2n+1) & \xrightarrow{\varphi} & \Lambda_{(m+1)}(2np-1) \to 0 \\ & & \downarrow H & & \downarrow \gamma_m \\ & & \Lambda_{(m)}(2np+2p^{m+1}-1) & = & \Lambda_{(m)}(2np+2p^{m+1}-1) \\ & & \downarrow & & \downarrow \\ & & 0 & & 0 \end{array}$$

Proof. $H' = \varphi|\Lambda_{(m)}(2n)$ (see 3.5), $H = \gamma_{m+1}\varphi_n$. All of the assertions are easy to check.

The EHP sequences easily give an EHP spectral sequence with an almost inductive procedure for computing, in the usual way. Corollary 3.6 calculates $H_*(\Lambda_{(m)}(0))$ which allows one to start an inductive calculation. One simple consequence of 4.2 and 3.6 is

Corollary 4.3. *$H : \widetilde{\Lambda_{(m)}}(1) \to \Lambda_{(m)}(2p^m - 1)$ is a chain equivalence, where $\widetilde{\Lambda_{(m)}}(1)$ is the connected cover (augmentation ideal) of $\Lambda_{(m)}(1)$.*

Applying this to (II) with $n = 0$ we get

Corollary 4.4. *There is a chain equivalence:*

$$\Lambda_{(m)}(2p^{m+1} + 1) \simeq \Lambda_{(m)}(2p^{m+1} - 1) \oplus \Lambda_{(m+1)}(2p^{m+2} - 1).$$

Restricting with (II) or (III) we get

Corollary 4.5. *There is a chain equivalence*

$$\Lambda_{(m)}(2p^{m+1}) \simeq \Lambda_{(m)}(2p^{m+1} - 1) \oplus \Lambda_{(m)}(2p^{m+2} - 1).$$

See also (6.9).

Finally from (II) we get

Corollary 4.6. *The inclusion $\Lambda_{(m)}(1) \to \Lambda_{(m+1)}(1)$ is null homotopic.*

5

In this section we will examine periodicity in this context. The connecting homomorphism from CMN (see 4.1) represents a compressed version of a v_{m+1} self-map. We wish to represent this as a chain map. For simplicity we replace m by $m - 1$.

Proposition 5.1. *Suppose $n \geq p^m$. Then there are chain maps of degree $q_m = 2p^m - 2$*

$$v_m : \Lambda_{(m-1)}(2n + 1) \to \Lambda_{(m-1)}(2n - 2p^m + 1) \subset \Lambda_{(m-1)}(2n + 1)$$

compatible for various n and inducing in homology the connecting homomorphism from 4.1.

Proof. A splitting of CNM from 4.1 is given as follows (with n decreased by $p^m - 1$):

$$e : \Lambda_{(m-1)}(2n + 1) \to \Lambda_{(m)}(2n - 2p^m + 1)$$

where $e(\tau_S \otimes \nu) = (-1)^{|S|}\tau(S \cup \{m\}) \otimes \nu$. e is not a chain map, but $\gamma e = 1$ and $\partial\gamma = -\gamma\partial$ (since γ has odd degree). In such a situation $\gamma(\partial e + e\partial) = 0$ so we may pull $e\partial + \partial e$ to $\Lambda_{(m-1)}(2n - 2p^m + 1)$. An explicit formula is easily obtained; viz.:

$$\begin{aligned} v_m(\tau_S \otimes \nu) = &- \sum_{T' \subset S} \tau((S, T) \cup \{m-1\}) \otimes \lambda_{p^T + p^{m-1}}\nu \\ &+ (-1)^{|S|} \sum_{\{0\} \cup T' \subset S} \tau((S - \{0\}, T) \cup \{m-1\}) \otimes \mu_{p^T + p^{m-1}}\nu \end{aligned}$$

if $m \geq 1$. $v_0 : \Lambda_{(-1)}(2n + 1) \to \Lambda_{(-1)}(2n - 1)$ is given by left multiplication by μ_0. These formulas are independent of n and hence compatible for various n.

Corollary 5.2. *Let* $v_m^{-1}\Lambda_{(m-1)}(2n+1)$ *be the telescope of the chain map*

$$v_m : \Lambda_{(m-1)}(2n+1) \to \Lambda_{(m-1)}(2n+1).$$

Then the suspension $\Lambda_{(m-1)}(2n-1) \xrightarrow{E^2} \Lambda_{(m-1)}(2n+1)$ *induces an isomorphism:*

$$v_m^{-1}\Lambda_{(m-1)}(2n-1) \xrightarrow{\cong} v_m^{-1}\Lambda_{(m-1)}(2n+1) \quad \text{for } n \geq 1$$

and the resulting periodic chain complex is isomorphic with $v_m^{-1}(HV(m-1)\tilde{\otimes}\Lambda)$.

Theorem 5.3. *If* $n > p^m$ *there is a commutative diagram:*

$$\begin{array}{ccc} \Lambda_{(m-1)}(2n+1) & \xrightarrow{\varphi} & \Lambda_{(m)}(2np-1) \\ \downarrow v_m & & \downarrow v_{m+1} \\ \Lambda_{(m-1)}(2n+1-2p^m) & \xrightarrow{\varphi} & \Lambda_{(m)}(2(n-p^m)p-1). \end{array}$$

In case $n = p^m$ *we have* $v_{m+1}\varphi_{p^m} - \varphi_0 v_m = e : \Lambda_{(m-1)}(2n+1) \to \Lambda_{(m)}(1)$ (*see* 5.1).

Proof. We calculate using the formulas of 3.5 and 5.1.

$$\begin{aligned} \varphi_{n-p^m} v_m(\tau_S \otimes \nu) &= (-1)^{|S|} \sum_{T' \subset S} \tau(S, T)'\tau_m \otimes H'_{n-p^m+p^{(S,T)}+p^{m-1}}(\lambda_{p^T+p^{m-1}}\nu) \\ &\quad - \sum_{T' \subset S} \tau_0\tau(S, T)'\tau_m \otimes H_{n-p^m+p^{(S,T)}+p^{m-1}}(\lambda_{p^T+p^{m-1}}\nu) \\ &\quad + \sum_{\{0\}\cup T' \subset S} \tau(S-\{0\}, T)'\tau_m \otimes H'_{n-p^m+p^{(S-\{0\},T)}+p^{m-1}}(\mu_{p^T+p^{m-1}}\nu) \\ &\quad + (-1)^{|S|} \sum_{\{0\}\cup T' \subset S} \tau_0\tau(S-\{0\}, T)'\tau_m \\ &\quad \otimes H_{n-p^m+p^{(S-\{0\},T)}+p^{m-1}}(\mu_{p^T+p^{m-1}}\nu) \end{aligned}$$

using 3.3, in case $n > p^m$ this reduces to

$$\begin{aligned} &(-1)^{|S|} \sum_{T' \subset S} \tau((S, T)' \cup \{m\}) \otimes (-\mu_{p^{T'}+p^m} H_{n+p^S}(\nu) - \lambda_{p^{T'}+p^m} H'_{n+p^S}(\nu)) \\ &\quad - \sum_{T' \subset S} \tau(\{0\} \cup (S, T)' \cup \{m\}) \otimes \lambda_{p^{T'}+p^m} H_{n+p^S}(\nu) \\ &\quad + (-1)^{|S|} \sum_{\{0\}\cup T' \subset S} \tau(\{0\} \cup (S-\{0\}, T)' \cup \{m\}) \\ &\quad \otimes (-\mu_{p^{T'}+p^{m+1}} H_{n+p^S}(\nu) - \lambda_{p^{T'}+p^{m+1}} H'_{n+p^S}(\nu)) \\ &= (-1)^{|S|+1} \sum_{T \subset S} \tau((S', T) \cup \{m\}) \otimes \lambda_{p^T+p^m} H'_{n+p^S}(\nu) \\ &\quad - \sum_{T \subset S} \tau((S' \cup \{0\}, T) \cup \{m\}) \otimes \lambda_{p^T+p^m} H_{n+p^S}(\nu) \\ &\quad + (-1)^{|S|+1} \sum_{T \subset S} \tau((S', T) \cup \{m\}) \otimes \mu_{p^T+p^m} H_{n+p^S}(\nu) \\ &= \nu_{m+1}\varphi(\tau_S \otimes \nu). \end{aligned}$$

In case $n = p^m$, the reduction of $\varphi_{n-p^m} v_m(\tau_S \otimes \nu)$ differs in the first and fourth terms. The terms corresponding to $T' = S$ and $\{0\} \cup T' = S$

reduce to $(-1)^{|S|}\tau(S\cup\{m\})\otimes\nu_1$. The corresponding terms in the expansion of $v_{m+1}\varphi(\tau_S\otimes\nu)$ are in the first and third sums with $T=S$. They yield

$$(-1)^{|S|+1}\tau(S\cup\{m\})\otimes\lambda_{n+p^s}H'_{n+p^s}(\nu)+\tau(S\cup\{m\})\otimes\mu_{n+p^s}H_{n+p^s}(\nu).$$

Thus $\varphi_{n-p^m}v_m(\tau_S\otimes\nu)-v_{m+1}\varphi(\tau_S\otimes\nu)=(-1)^{|S|}\tau(S\cup\{m\})\otimes\nu=e(\tau_S\otimes\nu)$ by 3.1.

6

In this section we show that there is a unique way to make $\Lambda_{(m)}$ into a differential algebra. Unstably this multiplication exactly reflects the composition theory in [G].

We give $\Lambda_{(m)}$ a bigrading by letting the s grading of $\tau_S\otimes\nu$ be the usual s grading of $\nu\in\Lambda$. Thus the boundary raises s by 1.

Theorem 6.1. *Suppose $\Lambda_{(m)}$ is a bigraded differential algebra such that $\Lambda_{(m-1)}$ is a subalgebra for $m\geq 0$ and if $s=0$ $\Lambda_{(m)}$ reduces to $E(\tau_0,\ldots,\tau_m)$. Then the following formulas determine the multiplication:*

$$\begin{aligned}
&\lambda_n\tau_0=\tau_0\lambda_n+\mu_n, \quad \mu_n\tau_0=\tau_0\mu_n,\\
&\lambda_n\tau_i=-\tau_i\lambda_n-\tau_{i-1}\lambda_{n+p^{i-1}}, \qquad i>0,\\
&\mu_n\tau_i=\tau_i\mu_n+\tau_{i-1}\mu_{n+p^{i-1}}, \qquad i>0.
\end{aligned}$$

Proof. The general method of proof will be to expand $\lambda_n\tau_i$ and differentiate the expansion. Using induction we then calculate the coefficients of the resulting over determined system. In some cases various terms in the expansion will be cycles (e.g. $\tau_k\lambda_{p^k}$), at which point the inductive step will need to be augmented.

Write $\lambda_n\tau_0=a_n\tau_0\lambda_n+b_n\mu_n$. Here the right-hand side is a general term of $\Lambda_{(0)}$ with $s=1$. Differentiate this equation with $n=1$ to obtain $b_1=1$. Differentiate this with $n=2$ and substitute in from the first equation to obtain $b_2=1$ and $a_1=a_2=-1$. Now apply induction after differentiating the general equation to conclude $a_n=-1$ and $b_n=1$. Expand $0=\lambda_n\tau_0^2$ to obtain $\mu_n\tau_0=\tau_0\mu_n$ for $n>0$. For $n=0$ differentiate $0=\tau_0^2$.

For an element in $\Lambda_{(m)}$ we will speak of the polynomial degree to mean the degree in the τ_i's. We claim now that $\lambda_n\tau_m$, when expanded, contains no terms of polynomial degree greater than one. This is proved by double induction on n and m. By induction, the expansion of $(\partial\lambda_n)\tau_m$ has degree one so the same is true for $\partial(\lambda_n\tau_m)$. Thus $\lambda_n\tau_m$ can have no term of degree bigger than 2 and the only possible degree 2 terms are $\tau_i\tau_k\nu_k$ where $\nu_k\in\Lambda$ is a cycle with $s=1$. Thus $\nu_k=\lambda_{p^t}$ and the stem degree of $\tau_i\tau_j\nu_k$ is congruent to 1 mod q. Since the stem degree of $\lambda_n\tau_m$ is congruent to 0 mod q, such terms cannot occur.

Now write $\lambda_n\tau_1=a_n\mu_{n+1}+b_n\tau_0\lambda_{n+1}+c_n\tau_1\lambda_n$. In case $n=1$ we differentiate to obtain $a_1=0$ and $b_1=c_1=-1$. Now apply $\lambda_n\tau_1\tau_0=-\lambda_n\tau_0\tau_1$ to obtain $\mu_n\tau_1=-b_n\tau_0\mu_{n+1}-c_n\tau_1\mu_n$. By induction assume $a_n=0$ and $b_n=c_n=-1$. Calculate $0=\lambda_n\tau_1^2$ to obtain $a_{n+1}=0$ and $c_{n+1}=-1$. Now differentiate and look at the terms of polynomial degree 0 to obtain $b_{n+1}=0$. Use $\lambda_n\tau_1\tau_0=-\lambda_n\tau_0\tau_1$ to obtain $\mu_n\tau_1=\tau_0\mu_{n+1}+\tau_1\mu_n$ for $n>0$. The case $n=0$ follows by differentiating $\tau_0\tau_1=-\tau_1\tau_0$.

We will simplify the general case by introducing derivations $\phi_m(x)=[\tau_m,x]=\tau_mx-(-1)^{|x|}x\tau_m$. We have $\phi_m(\phi_m(x))=0$ and $\phi_m(\phi_n(x))=-\phi_n(\phi_m(x))$.

Our calculations give $\phi_0(\mu_n) = 0$, $\phi_0(\lambda_n) = \mu_n$, $\phi_1(\mu_n) = -\tau_0\mu_{n+1}$ and $\phi_1(\lambda_n) = -\tau_0\lambda_{n+1}$.

Now differentiate $\tau_1\tau_m = -\tau_m\tau_1$ to get $0 = \tau_0(\lambda_1\tau_m + \tau_m\lambda_1 + \tau_{m-1}\lambda_{p^{m-1}+1}) = \tau_0(\phi_m(\lambda_1) + \tau_{m-1}\lambda_{p^{m-1}+1})$. Using $\phi_m(\phi_1(\lambda_n)) = -\phi_1(\phi_m(\lambda_n))$, $\phi_m(\tau_i) = 0$, and induction on n we get $0 = \tau_0(\phi_m(\lambda_n) + \tau_{m-1}\lambda_{p^{m-1}+n})$. Thus

$$\phi_m(\lambda_n) = -\tau_{m-1}\lambda_{p^{m-1}+n} + \alpha_n\tau_0\lambda_{n+p^{m-1}+\cdots+p+1}.$$

Assuming that $m > 1$, differentiating, and looking at the terms of polynomial degree 0 we get $0 = \alpha_n\mu_0\lambda_{n+p^{m-1}+\cdots+1}$. Thus $\alpha_n = 0$. Using $\phi_0(\phi_m(\lambda_n)) = -\phi_m(\phi_0(\lambda_n))$ we get $\phi_m(\mu_n) = -\tau_{m-1}\mu_{n+p^{m-1}}$.

We have next to show that the equations of 6.1 give $\Lambda_{(m)}$ the structure of an associative algebra. This will be done inductively using the following

Lemma 6.2. *Let Γ be a graded differetnial algebra and $\phi\colon \Gamma \to \Gamma$ be a derivation of odd degree with $\phi^2 = 0$. Then there is a differential algebra Γ' which is a free right Γ module with basis 1 and τ, and ϕ extends to a derivation ϕ' and Γ' by the formula $[\tau, \gamma] = \phi(\gamma)$.*

Proof. Use the formula $\tau\gamma = (-1)^{|\gamma|}\gamma\tau + \phi(\gamma)$ to define a multiplication extending the right Γ module structure. Thus

$$(a + \tau b)(c + \tau d) = (ac - (-1)^{|a|}\phi(a)d) + \tau(bc + (-1)^{|a|}ad - (-1)^{b}\phi(b)d).$$

Associativity is an easy verification, and the unit of Γ, if it has one, is a two-sided unit for Γ'.

As a simple application we may let $\Gamma = \Lambda$ and ϕ be the differential. We will write λ_0 for τ. This extends Λ to an associative algebra on λ_i, and μ_i for all $i \geq 0$. The usual formulas for the differential are then converted into the Adem relations, and it is a well-known observation that they then have the same form for all i. Let us call this algebra $E\Lambda$.

We will extend $E\Lambda_{(m)}$ to $E\Lambda_{(m+1)}$ by induction using 6.2. As in 6.1 we will need to consider the first two cases on their own right and then use general considerations to complete the inductive step.

Thus to construct $E\Lambda_{(0)}$, we need to construct a derivation ϕ_0 of $E\Lambda$ consistent with 6.1. To do this we must show that if we define ϕ_0 by $\phi_0(\lambda_n) = \mu_n$, $\phi_0(\mu_n) = 0$ and $\phi_0(ab) = \phi_0(a)b + (-1)^{|a|}a\phi_0(b)$, this defines a derivation of $E\Lambda$. We have precsribed an algorithm for calculation. We need to show that the Adem relations are carried into relations by this algorithm. With the notation of [HM], there are four kinds of Adem relations labeled (1.7), (1.8), (1.9) and (1.10). It is easy to see that (1.9) is carried to (1.10) and (1.10) is carried to $0 = 0$. Using the formula $d_{k,j} = c_{k-1,j-1} - c_{k,j}$ (in the notation of [HM]), one sees that (1.8) is carried to (1.10). The same formula is used to show that ϕ_0 carries (1.7) into (1.9) minus (1.8).

Thus $E\Lambda_{(0)}$ is an associative algebra. We write τ_0 for the introduced element and thus have formulas $\lambda_n\tau_0 = -\tau_0\lambda_m + \mu_n$ for all $n \geq 0$. In particular we define $d(x) = [\lambda_0, x]$ as before and have $d\tau_0 = \mu_0$. Thus with this differential we have recovered $\Lambda_{(0)}$.

We will prove by induction

Theorem 6.3. *$\Lambda_{(m)}$ has the structure of a differential algebra using the formulas of 6.1. If λ_0 is introduced, we have an algebra $E\Lambda_{(m)}$ with $d(x) = [\lambda_0, x]$.*

Proof. Having settled the case $m = 0$, we will assume, by induction that a multiplication in $\Lambda_{(m)}$ is constructed satisfying 6.1. One can easily add λ_0 using 6.2 so we have $E\Lambda_{(m)}$ with $d(x) = [\lambda_0, x]$. The induction will be complete if we can find a derivation $\phi_{m+1}: E\Lambda_{(m)} \to E\Lambda_{(m)}$ with $\phi_{m+1}(\tau_i) = 0$, $\phi_{m+1}(\lambda_n) = -\tau_m\lambda_{n+p^m}$ and $\phi_{m+1}(\mu_n) = -\tau_m\lambda_{n+p^m}$. Let A_m be the quotient of the free associative algebra generated by all λ_i, μ_i for $i \geq 0$ and τ_i for $0 \leq i \leq m$ modulo the relations:

(a) $\tau_i^2 = 0$, $\tau_i\tau_j = -\tau_j\tau_i$

(b) $\lambda_n\tau_0 = -\tau_0\lambda_n + \mu_n$, $\mu_n\tau_0 = \tau_0\mu_n$,

(c) $\lambda_n\tau_i = -\tau_i\lambda_n - \tau_{i-1}\lambda_{n+p^{i-1}}$,

(d) $\mu_n\tau_i = \tau_i\mu_n + \tau_{i-1}\mu_{n+p^{i-1}}$,

Now define a derivations $\rho : A_m \to A_m$ of degree q and $\phi_k : A_m \to A_m$ of degree $2(p^k - 1)$ for $0 \leq k \leq m+1$ by the rules:

$$\phi_k(\lambda_n) = -\tau_{k-1}\lambda_{n+p^{k-1}}, \qquad n \geq 0, k > 0,$$
$$\phi_k(\mu_n) = -\tau_{k-1}\mu_{n+p^{k-1}}, \qquad n \geq 0, k > 0,$$
$$\phi_0(\lambda_n) = \mu_n, \quad \phi_0(\mu_n) = 0, \quad \phi_k(\tau_i) = 0, \qquad k \geq 0,$$
$$\rho(\mu_n) = \mu_{n+1}, \quad \rho(\lambda_n) = \lambda_{n+1}, \quad \rho(\tau_i) = 0.$$

It is straightforward to check that these are well defined. The following relations hold:

$$(*) \qquad \phi_k\phi_l = -\phi_l\phi_k, \quad \phi_k\rho = \rho\phi_k.$$

Since these homomorphisms are derivations, it is only necessary to check equality on the generators.

$E\Lambda_{(m)}$ is a quotient of F_m by the relations (1.7) to (1.10) of [HM]. These derivations will define derivations of $E\Lambda_{(m)}$ if they preserve the two-sided ideal generated by these relations. We have already observed that ϕ_0 is defined on $E\Lambda_{(m)}$. The induction relies on

Lemma 6.4. *ρ defines a derivation $E\Lambda_{(m)} \to E\Lambda_{(m)}$.*

Continuing with the proof of 6.3 we note that all of the relations (1.7) and (1.10) can be generated by repeated application of ρ and ϕ_0 to $\lambda_i\lambda_{pi} = 0 = \mu_i\lambda_{pi+1}$. Indeed applying ρ to $\lambda_i\lambda_{pi} = 0$ k times yields

$$\sum_{s=0}^{k} \binom{k}{s} \lambda_{i+s}\lambda_{pi+k-s} = 0.$$

Using induction one can then write $\lambda_i\lambda_{pi+k}$ as a sum of admissible terms which yields (1.7). In the same manner $\mu_i\lambda_{pi+1} = 0$ leads to (1.9). Consequently we only need to show that $\phi_k(\lambda_i\lambda_{pi}) = 0 = \phi_k(\mu_i\lambda_{pi+1})$. This is straightforward.

Proof of 6.4. Using $\phi_0\rho = \rho\phi_0$ reduces the calculation to (1.7) and (1.9). Consider the relation (1.7):

$$\lambda_i\lambda_{pi+k} = \sum c_{k,j}\lambda_{i+k-j}\lambda_{pi+j}.$$

Applying ρ and expanding $\lambda_{i+1}\lambda_{pi+k}$, the formula reduces to

$$c_{k,j} + c_{k,j-1} - c_{k+1,j} = \begin{cases} c_{k-p,j-p} & \text{if } k \geq p, \\ 1 & \text{if } p > k = j, \\ 0 & \text{if } p > k > j. \end{cases}$$

This is easy to verify. The same congruence is used with (1.9) so ρ extends to a map of $E\Lambda_{(m)}$.

Proposition 6.5. *Suppose* $S = \{i_1, \dots, i_k\}$ *with* $i_1 < i_2 < \cdots < i_k$. *Then* $\tau_S = \tau_{i_1} \cdots \tau_{i_k}$.

Proof. Both sides of the equation have s filtrations 0 where there are no cycles if $S \neq \phi$. So it suffices to show that both sides have the same differential. Since the left-hand side is computed by 2.3 it suffices to compute the right side. Suppose $i_1 > 0$. We then prove by induction on k, that

$$\partial(\tau_{i_1} \cdots \tau_{i_k}) = (-1)^k \sum_{\substack{T' \subset S \\ T \neq \phi}} \tau(S, T) \otimes \lambda_{p^T}$$

in agreement with 2.3. The general case is obtained by calculating $\partial(\tau_0 \tau_S)$ with $0 \notin S$.

The following theorem suggests the compositions contemplated in [G].

Theorem 6.6. $\Lambda_{(m)}(n)_\sigma \Lambda_{(m)}(n+\sigma) \subset \Lambda_{(m)}(n)$.

The proof will be based on

Lemma 6.7.
(a) $\tau_S \Lambda_{(m)}(2n + 2p^S + 1) \subset \Lambda_{(m)}(2n+1)$,
(b) *if* $m \notin S$, $\tau_S \Lambda_{(m)}(2n + 2p^S) \subset \Lambda_{(m)}(2n)$,
(c) $\Lambda(2n+1)_\sigma \Lambda_{(m)}(2n + \sigma + 1) \subset \Lambda_{(m)}(2n+1)$,
(d) $\Lambda(2n+1)_\sigma \Lambda_{(m)}(2n+\sigma) \subset \Lambda_{(m)}(2n)$.

Proof of 6.6. Suppose $n = 2k+1$. Then

$$\begin{aligned}\Lambda_{(m)}(2k+1)_\sigma \Lambda_{(m)}(2k+\sigma+1) \\ &\subset \sum \tau_S \Lambda(2k + 2p^S + 1)_{\sigma - 2p^S + |S|} \Lambda_{(m)}(2k+\sigma+1) \\ &\subset \sum \tau_S \Lambda_{(m)}(2k + 2p^S + 1) \subset \Lambda_{(m)}(2k+1)\end{aligned}$$

by (c) and (a). Suppose now that $n = 2k$. We then have

$$\Lambda_{(m)}(2k)_\sigma \Lambda_{(m)}(2k+\sigma) \subset \sum \tau_S \Lambda(2k - 1 + 2p^S + \epsilon(S))_{\sigma - 2p^S + |S|} \Lambda_{(m)}(2k+\sigma).$$

We separate two cases. If $S = \{m\}$ we get

$$\begin{aligned}&\tau_m \Lambda(2k - 1 + 2p^m)_{\sigma - 2p^m + 1} \Lambda_{(m)}(2k+\sigma) \\ &\quad \subset \tau_m \Lambda_{(m)}(2k - 1 + 2p^m) \subset \Lambda_{(m)}(2n-1) \subset \Lambda_{(m)}(2k)\end{aligned}$$

by (c) and (a). If $S \neq \{m\}$. $\epsilon(S) + |S| \geq 2$ and we apply (d) to conclude that these terms are contained in

$$\sum \tau_S \Lambda_{(m)}(2k - 2 + 2p^S + \epsilon(S)).$$

If $m \in S$, $\epsilon(S) = 0$ and this is contained in $\Lambda_{(m)}(2k-1) \subset \Lambda_{(m)}(2k)$ by (a). If $m \notin S$, we apply (b) to see that such terms are contained in $\Lambda_{(m)}(2k)$.

For the proof of 6.7 we will need the following

Lemma 6.8. *For* $n \geq 0$ *and* $S \subset [m]$ *we have*

(a) $\lambda_n \tau_S = (-1)^{|S|} \sum_{T' \subset S} \tau(S, T)\lambda_{n+p^T} + \sum_{\{0\}\cup T' \subset S} \tau(S - \{0\}, T)\mu_{n+p^T}$

(b) $\mu_n \tau_S = \sum_{T' \subset S} \tau(S, T)\mu_{n+p^T}$.

Proof. (a) reduces to 2.3 if $n = 0$. Apply ρ n times to prove (a). Apply ϕ_0 to prove (b).

Proof of 6.7. (a) is immediate. To check (b) note that if $m \notin S$, $\epsilon(S \cup T) = \epsilon(T)$. We prove both (c) and (d) by induction on the s filtration in Λ. In case $s = 1$ we must prove

$$\begin{aligned} \lambda_k \Lambda_{(m)}(2n + kq - 1) &\subset \Lambda_{(m)}(2n), \\ \lambda_k \Lambda_{(m)}(2n + kq) &\subset \Lambda_{(m)}(2n + 1), \\ \mu_k \Lambda_{(m)}(2n + kq) &\subset \Lambda_{(m)}(2n), \\ \mu_k \Lambda_{(m)}(2n + kq + 1) &\subset \Lambda_{(m)}(2n + 1). \end{aligned}$$

Each of these are proven using 6.8. For example,

$$\begin{aligned} \lambda_k \Lambda_{(m)}(2n + kq - 1) &\subset \sum \lambda_k \tau_S \Lambda(2n + kq + 2p^S - 1) \\ &\subset \sum \tau(S, T)\lambda_{k+p^T} \Lambda(2n + kq + 2p^S - 1) \\ &\quad + \sum \tau(S - \{0\}, T)\mu_{k+p^T}\Lambda(2n + kq + 2p^S - 1). \end{aligned}$$

Suppose $S \neq T'$. Then $\lambda_{k+p^T}\Lambda(2n + kq + 2p^S - 1) \subset \Lambda(2n + 2p^{(S,T)} - 1)$ by 1.6, so these terms are contained in $\Lambda_{(m)}(2n - 1) \subset \Lambda_{(m)}(2n)$. If $S = T'$, $\lambda_{k+p^T}\Lambda(2n + kq + 2p^S) \subset \Lambda(2n + 2p^T)$ and $\tau_T \Lambda(2n + 2p^T) \subset \Lambda_{(m)}(2n)$ since $m \notin T$. A similar analysis applies to the second sum (where $T' \subset S - \{0\}$).

We prove (c) and (d) together by induction. Suppose that they are both valid for monomials of length $< s$ and let ν have length s where I is an admissible sequence. We distinguish two cases.

Case 1. $\nu_I = \lambda_i \nu_{I'} \in \Lambda(2n + 1)_\sigma$. Let $\sigma' = |\nu_{I'}| = \sigma - iq + 1$. Now $i \leq n$ so $\nu_{I'} \in \Lambda(2ip - 1)_\sigma \subset \Lambda(2n + iq - 1)_\sigma$. We can then use induction induction to see that:

$$\begin{aligned} \nu_{I'}\Lambda_{(m)}(2n + iq + \sigma' - 1) &\subset \Lambda_{(m)}(2n + iq - 1), \\ \nu_{I'}\Lambda_{(m)}(2n + iq + \sigma') &\subset \Lambda_{(m)}(2n + iq). \end{aligned}$$

Hence:

$$\begin{aligned} \nu_I \Lambda_{(m)}(2n + \sigma) &\subset \lambda_i \Lambda_{(m)}(2n + iq - 1) \subset \Lambda_{(m)}(2n), \\ \nu_I \Lambda_{(m)}(2n + \sigma + 1) &\subset \lambda_i \Lambda_{(m)}(2n + iq) \subset \Lambda_{(m)}(2n + 1). \end{aligned}$$

Case 2. $\nu_I = \mu_i \nu_{I'} \in \Lambda(2n + 1)_\sigma$. Let $\sigma' = |\nu_{I'}| = \sigma - iq$. Since $i \leq n$, $\nu_{I'} \in \Lambda(2ip + 1)_{\sigma'} \subset \Lambda(2n + iq + 1)_{\sigma'}$. Again by induction,

$$\begin{aligned} \nu_{I'}\Lambda_{(m)}(2n + iq + \sigma') &\subset \Lambda_{(m)}(2n + iq), \\ \nu_{I'}\Lambda_{(m)}(2n + iq + \sigma' + 1) &\subset \Lambda_{(m)}(2n + iq + 1). \end{aligned}$$

Then we get

$$\begin{aligned} \nu_I \Lambda_{(m)}(2n + \sigma) &\subset \mu_i \Lambda_{(m)}(2n + iq) \subset \Lambda_{(m)}(2n), \\ \nu_I \Lambda_{(m)}(2n + \sigma + 1) &\subset \mu_i \Lambda_{(m)}(2n + iq + 1) \subset \Lambda_{(m)}(2n + 1). \end{aligned}$$

This completes the induction and establishes 6.7.

One consequence of the existence of compositions is that elements of Hopf invariant one produce splittings. This gives us the following generalization of 4.5.

Proposition 6.9. *There is an isomorphism of chain complexes*

$$\Lambda_{(m)}(2p^k) \cong \Lambda_{(m)}(2p^k - 1) \oplus \Lambda_{(m)}(2p^{k+1} - 1).$$

Proof. $\lambda_{p^k} \in \Lambda(2p^k + 1) \subset \Lambda_{(m)}(2p^k)$ is a cycle. If $x \in \Lambda_{(m)}(2p^{k+1} - 1)$ then $\lambda_{p^k} x \in \Lambda_{(m)}(2p^k)$ by 6.6. Thus we may define a chain map

$$\eta : \Lambda_{(m)}(2p^{k+1} - 1) \to \Lambda_{(m)}(2p^k)$$

by $\eta(x) = \lambda_{p^k} x$.

Now write $x = \sum_{S\subset[m]} \tau_S \nu_S$ with $\nu_S \in \Lambda(2p^{k+1} + 2p^S - 1)$. Then we expand $\lambda_{p^k} x$ using 6.8a and observe that most terms are in $\Lambda_{(m)}(2p^k - 1)$. It is an easy matter to see that $H'_{p^k}(\eta(x)) = \varphi_{p^k}(\eta(x)) = x$. Thus the sequence

$$H' : 0 \to \Lambda_{(m)}(2p^k - 1) \to \Lambda_{(m)}(2p^k) \to \Lambda_{(m)}(2p^{k+1} - 1) \to 0$$

splits.

Finally, we introduce a Bockstein homomorphism into $\Lambda_{(m)}(n)$. Define β_m via the composites:

$$\begin{array}{ccccc}
\Lambda_{(m)}(2n+1) & \xrightarrow{\gamma_{m-1}} & \Lambda_{(m-1)}(2n+2p^m+1) & \xrightarrow{\iota} & \Lambda_{(m)}(2n+2p^m+1) \\
\uparrow E & & \uparrow E^2 & & \uparrow E^3 \\
\Lambda_{(m)}(2n) & \xrightarrow{\gamma_{m-1}} & \Lambda_{(m-1)}(2n+2p^m-1) & \xrightarrow{\iota_3} & \Lambda_{(m)}(2n+2p^m-2)
\end{array}$$

Theorem 6.10. *β_m is a derivation with $\beta_m^2 = 0$, $\beta_{m+1}\varphi = -\varphi\beta_m$ and $\beta_m\phi_{m+1} = -\rho^{p^m}$.*

Proof. As in the proof of 6.3, we construct a derivation $\beta_m : A_m \to A_m$ by $\beta_m(\tau_m) = 1$ and for $i < m$ $\beta(\tau_i) = 0$, $\beta(\lambda_n) = \beta(\mu_n) = 0$. One easily checks that $\beta_m = \iota\gamma_{m-1}$.

7

The complexes $\Lambda_{(m)}(n)$ have a unit $1 \in \Lambda(n) \subset \Lambda_{(m)}(n)$ and have their gradings normalized so that the unit has dimension 0. We will call this the stable grading. By way of contrast, the complexes in [BK] are graded according to their unstable dimension. We wish to regrade the complexes $\Lambda_{(m)}(n)$. Since they are not known to have a geometric analog, we seek an algebraic limitation on the regrading that reflects geometric considerations. The composition pairing of $\pi_\sigma(x)$ and $\pi_{\sigma+\tau}(S^\sigma)$ to $\pi_{\sigma+\tau}(x)$ suggests the following.

Definition 7.1. An unstable differential right Λ module (for short, unstable Λ module) is a graded chain complex $M = \{M_\sigma\}$ of $\mathbf{Z}_p$ vector spaces together with pairings $M_\sigma \otimes \Lambda(\sigma)_\tau \to M_{\sigma+\tau}$ which are associative and satisfy $m \cdot 1 = m$ and $d(m \cdot \lambda) = d(m) \cdot \lambda + (-1)^\sigma m \cdot d\lambda$ for $m \in M_\sigma$ and $\lambda \in \Lambda(\sigma)$.

Clearly the unstable cochain complex $M\tilde{\otimes}\Lambda$ of [BK] is an unstable Λ module. In fact, if we ignore the differential in $M\tilde{\otimes}\Lambda$ we see that it is the free unstable Λ module generated by the graded vector space M (in the sense of adjointness).

Given an unstable Λ module M we can construct ΩM by $(\Omega M)_\sigma = M_{\sigma+1}$. This will again be an unstable Λ module if we define the Λ action as follows. Write $\Omega x \in (\Omega M)_\sigma$ for the element corresponding to $x \in M_{\sigma+1}$. Then define a Λ module action by $(\Omega x)\lambda = (-1)^{|\lambda|}\Omega(x\lambda)$ and a derivation by $d(\Omega x) = \Omega(dx)$. Reversing this process (delooping) is in general, not possible. Let us consider, for example, regrading $\Lambda(n)$. We ask for which k is $\Omega^k\Lambda(n)$ an unstable Λ module,. For this we need $\Lambda(n)_{m+k}\cdot\Lambda(m) \subset \Lambda(n)$ which by [HM; 1.17] requires $m \le n+m+k$, i.e., $k \ge -n$. Thus $\Omega^{-n}\Lambda(n)$ is a delooping for $\Lambda(n)$. In general, $\Omega^k\Lambda(n)$ corresponds to $\Omega^{k+n}S^n$. If $n+1 \neq 2p^s$ for some s, $\Omega^{-n-1}\Lambda(n)$ is not an unstable Λ module. To see this, note that if it were, we could define $1\cdot\lambda_{n+1} \in (\Omega^{-n-1}\Lambda(n))_{(n+1)q-1} = \Lambda(n)_{(n+q)-1}$ with $d(1\cdot\lambda_{n+1}) = (-1)^{n+1}1\cdot d(\lambda_{n+1})$. Since $d(\lambda_{n+1}) \neq 0$ when $n+1 \neq 2p^s$ and is not a boundary in $\Lambda(n)$, this is impossible. Of course if $n+1 = 2p^s$, $\Lambda(n+1) \cong \Lambda(n) \oplus \Lambda(2p^{s+1}-1)$ so this is best possible.

Let $\widetilde{\Lambda(n)}$ be the augmentation ideal in $\Lambda(n)$. The next lemma implies that $\Omega^{-n-1}\widetilde{\Lambda(n)}$ is a right Λ module. This suggests that the fiber of the map $S^n \cup_{p\iota} e^{n+1} \to K(\mathbf{Z}_p, n)$ could be an H space (where $p > 2$).

Lemma 7.2. $\widetilde{\Lambda(n)}_d\Lambda(n+d+1) \subset \Lambda(n)$.

Proof. We show that $\nu_I\Lambda(n+d+1) \subset \Lambda(n)$ for $\nu_I \in \widetilde{\Lambda(n)}_d$ by induction on the length of I. First note that

$$\begin{aligned} &\lambda_k\Lambda(n+kq) \subset \Lambda(n) \quad \text{if } 2k \le n, \\ &\mu_k\Lambda(2m+kq+2) \subset \Lambda(2m+1) \quad \text{if } k \le m, \\ &\mu_k\Lambda(2m+kq+1) \subset \Lambda(2m) \quad \text{if } k < m. \end{aligned}$$

This handles the case $s = 1$. Suppose $\nu_I = \lambda_k\nu_{I'}$ with $2k \le n$. Then we have

$$\begin{aligned} \nu_I\Lambda(n+d+1) &\subset \lambda_k\widetilde{\Lambda(2kp-1)}_{d-kq+1}\Lambda(n+d+1) \\ &\subset \lambda_k\widetilde{\Lambda(n+kq-1)}_{d-kq+1}\Lambda(n+d+1) \\ &\subset \lambda_k\Lambda(n+kq-1) \subset \Lambda(n). \end{aligned}$$

On the other hand, suppose $\nu_I = \mu_k\nu_{I'}$ with $2k+1 \le n$. Then

$$\begin{aligned} \nu_I\Lambda(n+d+1) &\subset \mu_k\widetilde{\Lambda(2kp+1)}_{d-kq}\Lambda(n+d+1) \\ &\subset \mu_k\widetilde{\Lambda(n+kq)}_{d-kq}\Lambda(n+d+1) \\ &\subset \mu_k\Lambda(n+kq) \subset \Lambda(n). \end{aligned}$$

In the last step we apply the case $s = 1$.

We will call the delooping $\Omega^{-n}\Lambda(n)$ the unstable grading for $\Lambda(n)$ and distinguish it with the notation $\overline{\Lambda(n)}$. Thus $\Lambda(n) = \Omega^n\overline{\Lambda(n)}$. We wish to consider unstable gradings for the complexes $\Lambda_{(m)}(n)$. This should give us some insight as to how far the spaces considered in [G] could be delooped.

We first examine the classical EHP sequences in this light.

Definition 7.3. A map of unstable differential right Λ modules is a chain map of degree 0 which preserves the Λ module action.

We now claim that the following are exact sequences in the category of unstable Λ modules:

$$\begin{array}{ccccccccc} 0 & \to & \overline{\Lambda(2n-1)} & \xrightarrow{E} & \Omega\overline{\Lambda(2n)} & \xrightarrow{H'} & \Omega\overline{\Lambda(2np-1)} & \to & 0, \\ 0 & \to & \overline{\Lambda(2n)} & \xrightarrow{E} & \Omega\overline{\Lambda(2n+1)} & \xrightarrow{H} & \Omega\overline{\Lambda(2np+1)} & \to & 0. \end{array}$$

Furthermore neither H or H' can be delooped. The only hard part is to show that H and H' are maps of unstable Λ modules. This follows from the

Lemma 7.4. *Suppose* $x \in \Lambda(\widetilde{2n})_d$ *and* $y \in \Lambda(2n+d)$, *then* $H_n'(xy) = H_n'(x)y$. *Suppose* $x \in \Lambda(\widetilde{2n+1})_d$ *and* $y \in \Lambda(2n+d+1)$. *Then* $H_n(xy) = H_n(x)y$.

Note: This actually implies that the compositions

$$\Lambda(\widetilde{2n}) \to \Lambda(2n) \xrightarrow{H'} \Lambda(2np-1) \quad \text{and} \quad \Lambda(\widetilde{2n+1}) \to \Lambda(2n+1) \to \Lambda(2np+1)$$

are Λ module maps.

Proof. We only do the first case as the second is similar. Suppose first that $x \in \Lambda(\widetilde{2n-1})$. Then by 7.2, $xy \in \Lambda(2n-1)$ so the equation holds. Now the equation clearly holds when $x = \lambda_n$ so suppose that $x = \lambda_n \nu$ with $\nu \in \Lambda(\widetilde{2np-1})_{d-nq+1}$. Then, by the lemma, $\nu y \in \Lambda(2np-1)$ so $H_n'(xy) = \nu y = H_n'(x)y$.

We now ask, for which k is $\Omega^k \Lambda_{(m)}(n)$ an unstable Λ module?

Proposition 7.5. $\Omega^k \Lambda_{(m)}(n)$ *is a right* Λ *module if* $k \geq -n$, *but not if* $k = -n-1$ *where* $n+1 \neq 2p^s$ *or* $n + 2p^m \neq 2p^s$ *for some* $s \geq 0$.

Proof. Consider first $\Omega^k \Lambda_{(m)}(2n+1)$. The composition pairing must be defined on

$$\begin{aligned} [\Omega^k \Lambda_{(m)}(2n+1)]_d \otimes \Lambda(d) &= \Lambda_{(m)}(2n+1)_{d+k} \otimes \Lambda(d) \\ &= \sum \tau_S \Lambda(2n+1+2p^S)_{d+k-2p^S+|S|} \otimes \Lambda(d) \end{aligned}$$

for which we need $d \leq (2n+1+2p^S) + (d+k-2p^S+|S|)$; i.e., $2n+1+|S| \geq -k$. Thus $2n+1 \geq -k$ is sufficient. To see the converse, look at the term corresponding to $S = \phi$. In the other case we have:

$$\begin{aligned} [\Omega^k \Lambda_{(m)}(2n)]_d \otimes \Lambda(d) &= \Lambda_{(m)}(2n)_{d+k} \otimes \Lambda(d) \\ &= \sum \tau_S \Lambda(2n-1+2p^S+\epsilon_m(S))_{d+k-2p^S+|S|} \otimes \Lambda(d) \end{aligned}$$

and the requisite inequality is $-k \leq 2n-1+|S|+\epsilon_m(S)$. Since $|S|+\epsilon_m(S) \geq 1$, $-k \leq 2n$ suffices. For the converse directions look at terms where $S = \{m\}$.

This result supports the delooping conjecture of [G]: that $W_{(m)}^n$ deloops until it is $n-1$ connected, i.e., $W_{(m)}^n = \Omega^{m+1} V_{(m)}^n$. A particular case of importance is the space W_n which is the fiber of the double suspension. This suggests that W_n is a double loop space (at odd primes). At $p = 2$, the complex $\Lambda(W_n)$ (see [M]) only supports one delooping of W_n precisely because the extra terms in the boundary formula prevent it from being an unstable Λ module when delooped once more.

Theorem 7.5. *There are exact sequences in the category of unstable* Λ *modules*:

$$\begin{array}{lllllllll}
\overline{H} & : & 0 & \to & \overline{\Lambda_{(m)}(2n)} & \xrightarrow{E} & \Omega\overline{\Lambda_{(m)}(2n+1)} & \xrightarrow{H} & \Omega\overline{\Lambda_{(m)}(2np+2p^{m+1}-1)} & \to 0 \\
\overline{H}' & : & 0 & \to & \overline{\Lambda_{(m)}(2n-1)} & \xrightarrow{E} & \Omega\overline{\Lambda_{(m)}(2n)} & \xrightarrow{H'} & \Omega\overline{\Lambda_{(m)}(2np-1)} & \to 0 \\
\overline{E}^2 & : & 0 & \to & \overline{\Lambda_{(m)}(2n-1)} & \xrightarrow{E^2} & \Omega^2\overline{\Lambda_{(m)}(2n+1)} & \xrightarrow{\varphi} & \Omega\overline{\Lambda_{(m+1)}(2np-1)} & \to 0 \\
\overline{CMN} & : & 0 & \to & \overline{\Lambda_{(m)}(2n+1)} & \xrightarrow{\iota} & \overline{\Lambda_{(m+1)}(2n+1)} & \xrightarrow{\gamma} & \Omega\overline{\Lambda_{(m)}(2n+2p^{m+1}+1)} & \to 0 \\
\overline{RCMN} & : & 0 & \to & \Omega\overline{\Lambda_{(m)}(2n+1)} & \xrightarrow{\iota_3} & \overline{\Lambda_{(m+1)}(2n)} & \xrightarrow{\gamma} & \overline{\Lambda_{(m)}(2n+2p^{m+1}-1)} & \to 0
\end{array}$$

and a commutative diagram:

$$\begin{array}{ccc}
\Omega^2\overline{\Lambda_{(m)}(2n+2p^{m+1}-1)} & \longrightarrow & \Omega^2\overline{\Lambda_{(m)}(2n+1)} \\
{\scriptstyle E^2}\uparrow & \overset{v_{m+1}}{\searrow} & {\scriptstyle E^2}\uparrow \\
\Lambda_{(m)}\overline{(2n+2p^{m+1}-3)} & \longrightarrow & \overline{\Lambda_{(m)}(2n-1)}
\end{array}$$

where the horizontal maps are given by left composition with

$$v_{m+1}(1) \in \overline{\Lambda_{(m)}(k)}_{k+2p^{m+1}-2} \quad \textit{for } k \geq 1\,.$$

Proof. Since it is clear that the various inclusions are Λ module maps, one can reduce the question to checking γ, v_{m+1}, and φ. The first two are clear from the definition, while the third follows from 7.3.

The exact sequences $(*)$ actually split as right Λ modules (but not as chain complexes); we have

$$\begin{aligned}
\Omega\overline{\Lambda(2n+1)} &= \overline{\Lambda(2n)} \oplus \Omega\overline{\Lambda(2np+1)}, \\
\Omega\overline{\Lambda(2n)} &= \overline{\Lambda(2n-1)} \oplus \Omega\overline{\Lambda(2np-1)}.
\end{aligned}$$

By iteration we get

$$\Omega\Lambda(2n+1) = \bigoplus_{i=0}^{\infty} \mu_n\mu_{np}\dots\mu_{np^{i-1}}\Lambda(2np^i),$$

$$\Omega\Lambda(2n) = \Lambda(2n-1) \oplus \bigoplus_{i=0}^{\infty} \lambda_n\mu_{np-1}\dots\mu_{(np-1)p^i}\Lambda(2(np-1)p^i).$$

In particular, $\Omega\lambda(2n)$ and $\Omega\Lambda(2n+1)$ are free right Λ modules; consequently we have

Proposition 7.6. *If* M *is a free right* Λ *modules,* ΩM *is also free.*

It is possible to describe a basis for ΩM from a given basis $\{x_\alpha\}$ for M. For each x_α with $|x_\alpha| = 2n+1$, one takes $x_\alpha\mu_n\cdots\mu_{np^{i-1}}$ for each $i \geq 0$. For each x_α with $|x_\alpha| = 2n$ one takes x_α and $x_\alpha\lambda_n\mu_{np-1}\cdots\mu_{(np-1)p^{i-1}}$ for each $i \geq 0$. These basis elements can also be described in terms of homology operations. Using then notation of [C],

$$\Omega^{k+1}(M\widetilde{\otimes}\Lambda) = JD_k(L_k(M))\widetilde{\otimes}\Lambda.$$

In particular, we have the isomorphism

$$\overline{\Lambda_{(m)}(2n-1)} \cong \sum_{S\subset[m]} \Omega^{|S|}\overline{\Lambda(2n+2p^S-1)}$$

as right Λ modules. This is then a free right Λ module and suggests that the homology of the spaces $V_{(m)}(2n-1)$ of [G] should be

$$\bigotimes_{S\subset[m]} H_*(\Omega^{|S|}S^{2n+2p^S-1}).$$

8

In this section we describe in some detail an EHP spectral sequences for calculating $\mathrm{Ext}_A(H^*(V(m));\mathbf{Z}_p)$ for $-1 \le m \le \infty$. In the case $m=-1$ this method was used in [CGMM], and works just as well in the other cases. In particular, for $m=\infty$ it simplifies and can be used to calculate the cohomology of the subalgebra of the Steenrod algebra generated by the $\mathscr{P}^i$, $i \ge 0$.

The results are purely organizational. The short exact sequences H and H' give long exact sequences in homology and these fit together to form a spectral sequence. The s grading in the Λ algebra gives a bigrading to $\Lambda_{(m)}(n)$ and hence the spectral sequence is trigraded. We lay out the stem dimension (u) horizontally and the filtrations $v \ge 0$ vertically. We ignore s except when calculating differentials. It can be read off the genealogical description for reassembly into an Adams diagram. Write Λ^s_d for the subspace of the Λ algebra of elements of stem dimension d and monomial length s. We bigrade $\Lambda_{(m)}(n)$ by $\Lambda^s_{(m)}(n) = (\sum \tau_S \otimes \Lambda^s) \cap \Lambda_{(m)}(n)$. Then the differential in $\Lambda_{(m)}(n)$ increases s by 1.

Theorem 8.1. *For each $m \ge -1$ there is a trigraded spectral sequence $\{{}_sE^r_{u,v}\}$ with*

(a) $d^r : {}_sE^r_{u,v} \to {}_{s+1}E^r_{u-1,v-r}$,

(b) ${}_sE^1_{u,2v} = H_{u-qv+1}(\Lambda^{s-1}_{(m)}(2pv-1))$,
${}_sE^1_{u,2v+1} = H_{u-qv-q_{m+1}}(\Lambda^{s-1}_{(m)}(2pv+2p^{m+1}-1))$,

(c) $\mathrm{Ext}^{s,t}_A(H^*(V(m)), z_p) \cong \bigoplus_{i=0}^{\infty} {}_sE^\infty_{t-s,i}$,

(d) $H_u(\Lambda^s_{(m)}(v))$ *is the homology of the subspectral sequence with* ${}_tE^r_{u,v}=0$ *for* $t>s$,

(e)
$$ {}_sE^1_{u,0} = \begin{cases} \mathbf{Z}_p & \text{if } u=s=0, \\ 0 & \text{otherwise.} \end{cases}$$

The proof is straightforward. The case $m=\infty$ simplifies since half of the E^1 terms are missing.

Theorem 8.2. *There is a trigraded spectral sequence $\{{}_sE^r_{u,v}\}$ with*

(a) $d^r :{}_s E^r_{u,v} \to {}_{s+1}E^r_{u-1,v-r}$,

(b) ${}_sE^1_{u,v} = H_{u-qv+1}(\Lambda^{s-1}_{(\infty)}(2pv-1))$,

(c) $\mathrm{Ext}^{s,t}_A(H^*(V(\infty));\mathbf{Z}_p) \cong \mathrm{Ext}^{s,t}_P(\mathbf{Z}_p,\mathbf{Z}_p) \cong \bigoplus_{i=0}^{\infty} {}_sE^\infty_{t-s,t}$ *where $P\subset A$ is the subalgebra generated by the Steenrod pth powers,*

(d) $H_u(\Lambda^s_{(\infty)}(v)) = \bigoplus_{i=0}^{v} {}_sE^{\infty}_{u,i}$,

(e)

$$ {}_sE^1_{0,u} = \begin{cases} \mathbf{Z}_p & \text{if } u = s = 0, \\ 0 & \text{otherwise.} \end{cases} $$

Calculation with these spectral sequences is greatly aided by the following composition formula for Hopf invariants (see 7.4). Let $\widetilde{\Lambda_{(m)}}(n)$ be the augmentation ideal.

Theorem 8.3. *Suppose* $x \in \widetilde{\Lambda_{(m)}(2n)}_d$ *and* $y \in \Lambda_{(m)}(2n+d)$ *then* $H'_n(xy) = H'_n(x)y$. *Suppose* $x \in \Lambda_{(m)}(\widetilde{2n}+1)_d$ *and* $y \in \Lambda_{(m)}(2n+d+1)$. *Then* $H_n(xy) = H_n(x)y$.

Thus the Hopf invariant of a composition is always obtained by taking the Hopf invariant of the first factor and multiplying by the second factor, provided the first factor is of positive stem degree.

The proof of 8.3 is long but entirely analogous to that of 7.4. It is accomplished by showing first that $\varphi(xy) = \varphi(x)\iota(y)$ for $x \in \Lambda_{(m)}(2n+1)_d$ and $y \in \Lambda_{(m)}(2n+d+1)$. The Hopf invariants are easily derived from φ. The result is then seen to follow from the analog of 7.2; namely

$$ \Lambda_{(m)}(\widetilde{2n}+1)_d \Lambda_{(m)}(2n+d+2) \subset \Lambda_{(m)}(2n+1). $$

This is obtained by a modification of the analysis in §6.

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